Assignment 1.

This assignment is due February 14th. If you need more time, ask for an extension (just don't get overwhelmed by homeworks piling up.)

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

(1) If

$$A = \begin{pmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

find all solutions of $AX = 2X$ and all solutions of $AX = 3X$. (The symbol cX denotes the column each entry of which is c times the corresponding entry of X.)

(2) Find a row-reduced matrix which is row-equivalent to

$$A = \begin{pmatrix} i & -(1+i) & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix}$$

(3) Prove that the following two matrices are *not* row-equivalent:

$$\begin{pmatrix} 2 & 0 & 0 \\ a & -1 & 0 \\ b & c & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 2 \\ -2 & 0 & -1 \\ 1 & 3 & 5 \end{pmatrix}$$

(4) Give an example of a system of two linear equations in two unknowns which has no solution.

(5) Let

$$A = \begin{pmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{pmatrix}.$$

For which columns (y_1, y_2, y_3, y_4) does the system $AX = Y$ have a solution?

(6) If $\mathbb{C}$ is field of complex numbers, which vectors in $\mathbb{C}^3$ are linear combinations of $(1, 0, -1)$, $(0, 1, 1)$, and $(1, 1, 1)$?

(7) Let V be the set of all pairs (x, y) of real numbers, and let F be the field of real numbers. Define

$$(x, y) + (x_1, y_1) = (x + x_1, y + y_1),$$

$$c(x, y) = (cx, y).$$

Is V, with these operations, a vector space over the field of real numbers?

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(8) On $\mathbb{R}^n$, define two operations

$$\alpha \oplus \beta = \alpha - \beta,$$
$$c \odot \alpha = -c\alpha.$$

The operations on the right are the usual ones. Which of axioms for a vector space are satisfied by $(\mathbb{R}^n, \mathbb{R}, \oplus, \odot)$?

(9) Let V be the set of pairs (x, y) of real numbers and let F be the field of real numbers. Define

$$(x, y) + (x_1, y_1) = (x + x_1, 0),$$
$$c(x, y) = (cx, 0).$$

Is V, with these operations, a vector space?

(10) Which of the following sets of vectors $\alpha = (a_1, a_2, \dots, a_n)$ in $\mathbb{R}^n$ are subspaces of $\mathbb{R}^n$ $(n \geq 3)$?
(a) all α such that $a_1 \geq 0$;
(b) all α such that $a_1 + 3a_2 = a_3$;
(c) all α such that $a_2 = a_1^2$;
(d) all α such that $a_1 a_2 = 0$;
(e) all α such that a_2 is rational.

(11) Let V be the (real) vector space of all functions f from $\mathbb{R}$ to $\mathbb{R}$. Which of the following sets of functions are subspaces of V?
(a) all f such that $f(x^2) = f(x)^2$;
(b) all f such that $f(0) = f(1)$;
(c) all f such that $f(3) = 1 + f(-5)$;
(d) all f such that $f(-1) = 0$;
(e) all f which are continuous.

(12) Let W_1 and W_2 be subspaces of a vector space V such that the set-theoretic union of W_1 and W_2 is also a subspace. Prove that one of the spaces W_i is contained in the other.

(13) Let W_1 and W_2 be subspaces of a vector space V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$. Prove that for each vector α in V there are *unique* vectors $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$ such that $\alpha = \alpha_1 + \alpha_2$.